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Unsteady lifting-line theory as a singular-perturbation problem

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Unsteady lifting-line theory is developed for a wing of large aspect ratio oscillating at low frequency in inviscid incompressible flow. The wing is assumed to have a rigid chord but a flexible span. Use of the method of matched asymptotic expansions reduces the problem from a singular integral equation to quadrature. The pressure field and airloads, for a prescribed wing shape and motion, are obtained in closed form as expansions in inverse aspect ratio. A rigorous definition of unsteady induced downwash is also obtained. Numerical calculations are presented for an elliptic wing in pitch and heave; compared with numerical lifting-surface theory, computation time is reduced significantly. The present work also identifies and resolves errors in the unsteady lifting-line theory of James (1975), and points out a limitation in that of Van Holten (1975, 1976, 1977).

1. Introduction

Important unsteady and three-dimensional effects occur for a wide range of problems of practical interest involving oscillating flexible wings of large aspect ratio. Many of these effects cannot be predicted accurately by strip-theory or quasi-steady aerodynamics. The cost of numerical implementation of current unsteady lifting-surface theory, the non-analytic nature of the results, and the success of Prandtl's lifting-line theory for steady flow have prompted several recent investigations that extend the concepts of lifting-line theory to unsteady flows. These studies have employed the method of matched asymptotic expansions (as, for steady flow, did Van Dyke 1963), and have been termed 'unsteady lifting-line theory'. Unfortunately, some of the existing unsteady lifting-line theories (for incompressible flow) are incomplete or incorrect, as can be seen from the following discussion.

The unsteady lifting-line theory of James (1975), for a straight flexible wing in general unsteady motion, is based on a semi-intuitive matched asymptotic expansions (MAE) approach. The present work shows that his unsteady induced downwash contains an unremoved logarithmic singularity and is thus infinite, so that his three-dimensional results are incorrect. Although the theory is said to be valid for all reduced frequencies, at several points the formulation assumes low reduced frequencies. He also assumes an inner solution for the acceleration potential $O(A^{-1})$, whereas, because of the independence of scale in inviscid flows, it must be $O(1)$. The inner solution also lacks the eigensolutions arising from the lack of boundary conditions at infinity on the scale of the inner region. Furthermore, this work does not treat and resolve the inherent non-uniqueness of the solution in the acceleration potential formulation of the problem. These problems are resolved in the present paper.

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Van Holten (1975, 1976, 1977) has developed lifting-line theories for a rigid rectangular wing in uniform motion, with and without yaw and harmonic pitching motion, and also as a helicopter rotor blade in forward flight. Van Holten, like James, assumes that induced downwash is constant across the chord. It is shown in the present work that this is true only for the lowest-order lifting-line theory for small reduced frequencies, while, as was first pointed out by Reissner (1944), for a harmonically oscillating wing, induced downwash has a sinusoidal variation across the chord. Also, the second-order steady lifting-line theory of Van Dyke (1963) shows that to second order steady induced downwash varies linearly across the chord. Van Holten also regards his work as valid for all reduced frequencies, whereas his formulation, like that of James, is limited to low reduced frequencies.

Van Holten (1976) was the first to point out the correct interpretation of induced downwash in steady and unsteady flows. The same interpretation comes out of the present work.

The unsteady lifting-line theory of Cheng (1975) treats a wing with curved and swept planform in harmonic oscillation. This work is incomplete in that it does not include calculation of the aerodynamic loads, unsteady induced downwash for the low-frequency domain, and an element of the inner solution (particular solution for the non-homogeneous trailing-vortex-sheet boundary condition). These problems, however, have been resolved in later works (see e.g. Murillo 1979; Cheng & Murillo 1984).

The problem of a harmonically oscillating finite wing involves three characteristic lengthscales: chord c, span b and wavelength $\lambda = 2\pi U/\omega$ of the periodic wake, where U is the free-stream velocity and ω is the radian frequency of oscillation. To characterize the influence of unsteadiness on three-dimensional effects, Cheng (1975) has identified five ranges of λ for a high-aspect-ratio wing ($c \ll b$):

I $c \ll b \ll \lambda$, very low frequency;

II $c \ll b = O(\lambda)$, low frequency;

III $c \ll \lambda \ll b$, intermediate frequency;

IV $c = O(\lambda) \ll b$, high frequency;

V $\lambda \ll c \ll b$, very high frequency.

Domain I corresponds to very low frequencies where quasi-steady aerodynamic theory is adequate. Domain V, on the other hand, corresponds to very high frequencies where self-averaging of the high-frequency periodic wake renders the problem locally two-dimensional. In domain II the reduced frequency based on the span is $\omega b/U = O(1)$, whereas in domain IV the reduced frequency based on the chord is $\omega c/U = O(1)$. (Guiraud & Slama (1981) have developed a high-frequency unsteady lifting-line theory. They find that the leading three-dimensional correction is $O(A^{-2} \log A)$, which is consistent with Cheng's conclusions.) The analysis of the problem in domains II and IV involves two distinct regions in space, corresponding to lengthscales c and b, whereas the analysis of domain III involves three regions in space, corresponding to c, b and λ.

This paper is devoted to the development of an unsteady lifting-line theory valid in domains I and II. The theory is formulated in terms of the acceleration potential ψ. The advantages of this formulation are that ψ is continuous everywhere except across the wing, and the pressure distribution on the wing is obtained directly from ψ. However, the solution is not unique, since multiples of eigensolutions with

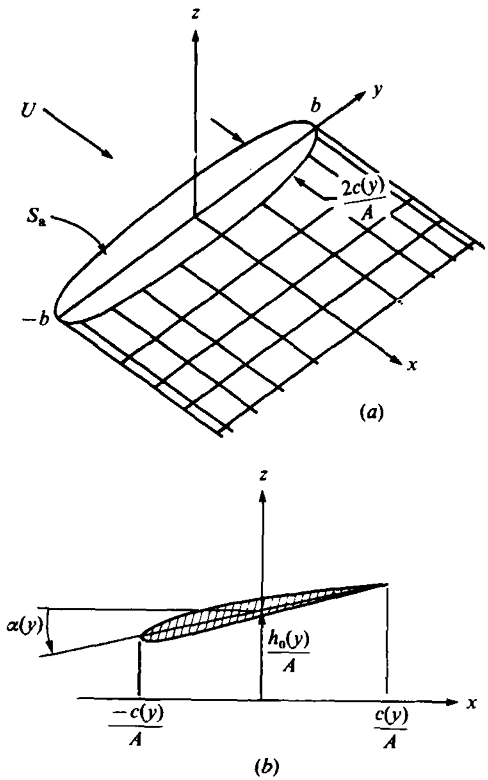

FIGURE 1. (*a*) Schematic of the wing in unsteady motion; (*b*) positive direction of pitch and heave for wing sections.

$\partial\psi/\partial z = 0$ at the wing may be added. Uniqueness is achieved by determining the downwash by integration of ψ from far upstream to some point on the wing.

The present lifting-line theory is developed for a wing with straight unswept meanline in incompressible flow. The effects of meanline sweep and curvature are accounted for by Murillo (1979). Compressibility effects are accounted for by Cheng & Meng (1980).

2. Problem formulation

Consider a thin, unswept, almost-planar wing of large aspect ratio, executing small-amplitude harmonic oscillations normal to the wing planform in a uniform stream of inviscid incompressible fluid with velocity U directed along the x-axis. The wing planform about the midchord is described by

$$x = \pm\frac{c(y)}{A} \quad (|y| \leqslant b,\ z = 0) \tag{2.1}$$

in a Cartesian coordinate system (x, y, z) fixed to the mean position of the wing (see figure 1*a*). A is the wing aspect ratio defined as $A = (2b)^2/S_a$, where b is the semispan and S_a is the wing planform area. $c(y)/A$ is the semichord. Both b and $c(y)$ are $O(1)$.

The transverse displacements of the wing are described by

$$z = h(x,y,t) = \left[\frac{h_0(y)}{A} + \alpha(y)\,x\right] e^{j\omega t}$$
$$= \left\{\frac{1}{2}\frac{c_0}{A}\,\xi_0(y) + [\xi_1(y) + j\xi_2(y)]\,x\right\} e^{j\omega t} \quad \left(|x| \leqslant \frac{c(y)}{A}, \quad |y| \leqslant b\right), \tag{2.2}$$

where ξ_0, ξ_1 and ξ_2 are the non-dimensional heave/pitch amplitudes, c_0/A is the root semichord, j is the temporal complex unit, ω is the radian frequency of oscillation and t is time. It is assumed that $\omega c(y)/U = O(1)$. Equations (2.2) define a spanwise flexible wing executing arbitrary torsional and bending oscillations. The heaving motion is positive in the z-direction and the pitching motion is positive nose-down (see figure 1*b*).

We require that the functions h_0 and α satisfy the conditions of the linearized theory, so that

$$\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}, U^{-1}\frac{\partial h}{\partial t} \ll 1.$$

Implicit in the choice of $h(x,y,t)$ and $c(y)$ is the fundamental assumption of lifting-line theory that spanwise flow perturbations are small compared with those in planes normal to the span. Difficulties arising from blunt wingtips are discussed in §8.

The above problem is formulated in terms of the acceleration potential $\psi(\boldsymbol{x},t) = [p_\infty - p(\boldsymbol{x},t)]/\rho$, where $\boldsymbol{x} = (x,y,z)$, p is pressure, ρ is fluid density and p_∞ is the free-stream pressure. ψ is governed by the following boundary-value problem:

$$\nabla_3^2\,\psi(\boldsymbol{x},t) = 0, \tag{2.3a}$$

$$\frac{\partial}{\partial z}\psi(\boldsymbol{x},t) = \frac{\mathrm{D}}{\mathrm{D}t}\,W_0(x,y,t) = \frac{\mathrm{D}^2}{\mathrm{D}t^2}\,h(x,y,t) \quad \left(|x| \leqslant \frac{c(y)}{A}, \quad |y| \leqslant b, \quad z = 0\pm\right), \tag{2.3b}$$

$$|\psi(\boldsymbol{x},t)| < \infty \quad \left(x = \frac{c(y)}{A}, \quad |y| \leqslant b, \quad z = 0\right), \tag{2.3c}$$

$$\psi(\boldsymbol{x},t) = 0 \quad \left(|x| > \frac{c(y)}{A}, \quad z = 0\right), \tag{2.3d}$$

$$\psi(\boldsymbol{x},t) \to 0 \quad (|\boldsymbol{x}| \to \infty), \tag{2.3e}$$

where ∇_3^2 is the three-dimensional Laplacian operator, $\mathrm{D}/\mathrm{D}t = \partial/\partial t + U\partial/\partial x$ is the linearized substantial derivative and $W_0(x,y,t) = \mathrm{D}h(x,y,t)/\mathrm{D}t$ is the linearized downwash at the wing.

The solution of this problem can be expressed in terms of a distribution of pressure doublets over the projection of the wing planform on the (x,y)-plane (Küssner 1941):

$$\tilde{\psi}(\boldsymbol{x}) = \frac{-1}{4\pi\rho}\int_{-b}^{b} \mathrm{d}\eta \int_{-c(y)/A}^{c(y)/A} \mathrm{d}\xi\,\Delta\tilde{p}(\xi,\eta)\frac{\partial}{\partial z}\frac{1}{R}, \tag{2.4}$$

where $R = [(x-\xi)^2 + (y-\eta)^2 + z^2]^{\frac{1}{2}}$ and $\Delta p = p_\ell - p_u$ is the local pressure jump across the wing. $(\ )_u$ and $(\ )_\ell$ denote the upper and lower wing surfaces respectively, and $(\tilde{\ })$ denotes the complex amplitude of harmonic functions.

To construct the asymptotic solution, we consider two simplified limits of the problem as $A \to \infty$: the outer limit and the inner limit. The outer limit corresponds to fixed span; as $A \to \infty$ the chord tends to zero and the wing collapses to a loaded line. The inner limit corresponds to fixed chord; as $A \to \infty$ the span tends to infinity. The outer and inner limits are both both incomplete representations of the full problem, each lacking some essential features of the problem: details of the airfoil in the outer limit and the three-dimensional effects in the inner limit. Matching the two expansions resolves this indeterminacy. However, the solution to (2.3*a*–*e*) is not unique, because multiples of eigensolutions with $\partial\psi/\partial z = 0$ at the wing may be present. Uniqueness is achieved by determining the downwash by integration of ψ from far upstream to some point on the wing.

3. Outer solution for the acceleration potential

Here we week an expansion for ψ in the outer region (distances from the wing of order of span) where the wing shrinks to a loaded line as $A \to \infty$. This is obtained from (2.4) by expanding R^{-1} for small ξ and integrating across the chord. The three-term outer expansion is

$$\tilde{\psi}^o(x) \sim \frac{-1}{4\pi\rho}\left\{\frac{\partial}{\partial z}\int_{-b}^{b}\frac{\tilde{l}(\eta)}{R_0}\,d\eta + \frac{\partial^2}{\partial x\,\partial z}\int_{-b}^{b}\frac{\tilde{m}(\eta)}{R_0}\,d\eta + \frac{1}{2}\frac{\partial^3}{\partial x^2\,\partial z}\int_{-b}^{b}\frac{\tilde{q}(\eta)}{R_0}\,d\eta + \mathrm{HOT}\right\}, \tag{3.1}$$

where $R_0 = [x^2+(y-\eta)^2+z^2]^{\frac{1}{2}}$, $(\ )^o$ denotes the outer region, HOT denotes higher-order terms, $\tilde{l}(y) = O(A^{-1})$ is section lift, $\tilde{m}(y) = O(A^{-2})$, and

$$\tilde{m}(y) = -\int_{-c(y)/A}^{c(y)/A} \xi\,\Delta\tilde{p}(\xi, y)\,d\xi = O(A^{-2}),$$

$$\tilde{q}(y) = \int_{-c(y)/A}^{c(y)/A} \xi^2\,\Delta\tilde{p}(\xi, y)\,d\xi = O(A^{-3})$$

are respectively the first and second moments of section lift about the midchord. $\tilde{m}$ and $\tilde{q}$ are respectively positive in the clockwise and counterclockwise directions. It is seen that the outer expansion consists of spanwise distributions of multipoles along the loaded line. The first term consists of dipoles of strength $\tilde{l}(y)$, the second term consists of quadrupoles of strength $\tilde{m}(y)$, and so on. The above outer expansion is in agreement with that of James (1975), who gives the first two terms of (3.1).

3.1. *Inner expansion of outer expansion*

The inner expansion of the outer expansion is obtained from (3.1) in the limit of $r = (x^2+z^2)^{\frac{1}{2}} \to 0$. In terms of the magnified (inner) variables, $\hat{x} = Ax = \hat{r}\cos\theta$ and $\hat{z} = Az = \hat{r}\sin\theta$, the inner expansion of the three-term outer expansion is

$$\begin{aligned}
\tilde{\psi}^{oi}(x) \sim \frac{1}{2\pi\rho}\Bigg\{ & A\,\tilde{l}(y)\,\frac{\sin\theta}{\hat{r}} + \frac{1}{4A}\,\hat{r}\sin\theta \\
& \times\left[\frac{\partial^3}{\partial y^3}\int_{-b}^{b}\tilde{l}(\eta)\log|y-\eta|\,\mathrm{sgn}\,(y-\eta)\,d\eta \right. \\
& \left. +\left(1+2\log\frac{2A}{\hat{r}}\right)\tilde{l}''(y)\right] + O(A^{-3}\tilde{l}) \quad \text{dipole} \\
& -A^2\tilde{m}(y)\,\frac{\sin 2\theta}{\hat{r}^2} \\
& -\tfrac{1}{4}\tilde{m}''(y)\sin 2\theta + O(A^{-2}\tilde{m}) \quad \text{quadrupole} \\
& +A^3\tilde{q}(y)\,\frac{\sin 3\theta}{\hat{r}^3} \\
& -\tfrac{1}{8}A\tilde{q}''(y)\,\frac{\sin\theta - \sin 3\theta}{\hat{r}} \quad \text{octupole} \\
& +O(A^{-1}\tilde{q}) \\
& +\mathrm{HOT}\Bigg\} \quad (A\to\infty,\ \hat{r} = O(1),\ |y| \leqslant b),
\end{aligned} \tag{3.2}$$

where $(\ )'$ denotes derivative with respect to the indicated argument. The terms in the expansion are grouped together so as to identify the inner expansion of each term of the outer expansion (3.1). It is seen that the spanwise distribution of each multipole in the outer expansion reduces to a two-dimensional multipole of the same order plus higher-order terms representing three-dimensional correction. James (1975) obtained the first term of the dipole and the quadrupole expansions. Except for a missing factor of A (apparently a misprint), his result is in agreement with (3.2).

4. Inner solution and eigensolutions for the acceleration potential

To determine the flow near the wing, we magnify the cross-sectional coordinates so that the two-dimensional airfoil character of the flow is obtained in the limit of $A \to \infty$. Thus

$$\hat{x} = Ax, \quad \hat{z} = Az. \tag{4.1}$$

In the boundary-value problem at hand, time enters in only through the boundary condition at the wing, which, in terms of the inner variables, is given by

$$W_0 = \left(\frac{1}{A}\frac{\partial}{\partial t} + U\frac{\partial}{\partial \hat{x}}\right)\hat{h} = W_0^{\mathrm{i}} \quad (|\hat{x}| \leqslant c(y),\ |y| \leqslant b,\ \hat{z} = 0\pm), \tag{4.2}$$

where $\hat{h} = Ah$, and $(\ )^{\mathrm{i}}$ denotes the inner region. In the inner problem A and t always appear in the combination At, which we will denote by $\hat{t}$.

We assume that, in the inner region, the acceleration potential may be expanded in an asymptotic series in inverse aspect ratio of the form

$$\psi^{\mathrm{i}}(\hat{\boldsymbol{x}},\hat{t}) \sim \psi_0^{\mathrm{i}}(\hat{\boldsymbol{x}},\hat{t}) + A^{-1}\psi_1^{\mathrm{i}}(\hat{\boldsymbol{x}},\hat{t}) + A^{-2}\log A\,\psi_2^{\mathrm{i}}(\hat{\boldsymbol{x}},\hat{t}) + A^{-2}\psi_3^{\mathrm{i}}(\hat{\boldsymbol{x}},\hat{t}) + \dots \quad (A \to \infty), \tag{4.3}$$

where $\hat{\boldsymbol{x}} = (\hat{x}, y, \hat{z})$. Since physical quantitites are independent of scale in inviscid flows (Ashley & Landahl 1965), the first term of the expression is $O(1)$. James's (1975) expansion has a leading term $O(A^{-1})$, which is incorrect. We include logarithmic terms in (4.3) because of the anticipated matching to logarithmic terms in ψ^{oi}, (3.2). Matching will show that the first two terms of (4.3) do not contain logarithms.

Introducing (4.1)–(4.3) into the full problem we obtain a series of simplified problems for ψ_n^{i}. The lowest-order inner solution ψ_0^{i} satisfies the following boundary-value problem:

$$\hat{\nabla}_2^2\psi_0^{\mathrm{i}}(\hat{\boldsymbol{x}},\hat{t}) = \left(\frac{\partial^2}{\partial \hat{x}^2} + \frac{\partial^2}{\partial \hat{z}^2}\right)\psi_0^{\mathrm{i}} = 0, \tag{4.4a}$$

$$\frac{\partial}{\partial \hat{z}}\psi_0^{\mathrm{i}}(\hat{\boldsymbol{x}},\hat{t}) = \frac{\mathrm{D}}{\mathrm{D}\hat{t}}W_0^{\mathrm{i}}(\hat{x},y,\hat{t}) \quad (|\hat{x}| \leqslant c(y),\ |y| \leqslant b,\ \hat{z} = 0\pm), \tag{4.4b}$$

$$|\psi_0^{\mathrm{i}}(\hat{\boldsymbol{x}},\hat{t})| < \infty \quad (\hat{x} = c(y),\ |y| \leqslant b,\ \hat{z} = 0), \tag{4.4c}$$

$$\psi_0^{\mathrm{i}}(\hat{\boldsymbol{x}},\hat{t}) = 0 \quad (|\hat{x}| > c(y),\ \hat{z} = 0), \tag{4.4d}$$

$$\psi_0^{\mathrm{i}}(\hat{\boldsymbol{x}},\hat{t}) \to ? \quad (\hat{r} \to \infty), \tag{4.4e}$$

where $\mathrm{D}/\mathrm{D}\hat{t} = \partial/\partial\hat{t} + U\partial/\partial\hat{x}$, $\hat{r} = (\hat{x}^2+\hat{z}^2)^{\frac{1}{2}}$, and $\hat{\nabla}_2^2$ denotes the two-dimensional Laplacian operator. The main simplification here is the reduction of the three-dimensional Laplace equation to a two-dimensional one. We choose ψ_0^{i} to satisfy the total downwash boundary condition at the wing W_0^{i} to all orders. This makes ψ_0^{i} the exact unsteady airfoil solution $\psi_{2\mathrm{D}}^{\mathrm{i}}$, which is the dominant feature of the inner solution. The higher-order terms in ψ^{i} then satisfy homogeneous boundary conditions at the wing surface. The loss of boundary conditions at infinity due to the stretching of the variables implies the presence of eigensolutions in the solution. These satisfy homogeneous boundary conditions, but may not vanish at infinity. Hence ψ_0^{i} consists of $\psi_{2\mathrm{D}}^{\mathrm{i}}$ and multiples of these eigensolutions.

The boundary-value problem governing ψ_1^{i} and ψ_2^{i} is

$$\hat{\nabla}_2^2\,\psi^{\mathrm{i}}(\hat{\boldsymbol{x}},\hat{t}) = 0, \tag{4.5a}$$

$$\frac{\partial\psi^{\mathrm{i}}}{\partial\hat{z}}(\hat{\boldsymbol{x}},\hat{t}) = 0 \quad (|\hat{x}| \leqslant c(y),\ |y| \leqslant b,\ \hat{z} = 0\pm), \tag{4.5b}$$

$$\psi^{\mathrm{i}}(\hat{\boldsymbol{x}},\hat{t}) = 0 \quad (|\hat{x}| > c(y),\ \hat{z} = 0), \tag{4.5c}$$

$$|\psi^{\mathrm{i}}(\hat{\boldsymbol{x}},\hat{t})| < \infty \quad (\hat{x} = c(y),\ |y| \leqslant b,\ \hat{z} = 0), \tag{4.5d}$$

$$\psi^{\mathrm{i}}(\hat{\boldsymbol{x}},\hat{t}) \to\ ? \quad (\hat{r}\to\infty). \tag{4.5e}$$

The solution of this homogeneous boundary-value problem consists of eigensolutions alone.

To determine the solution of (4.4*a–e*), we notice that, with the additional boundary condition $\psi_0^{\mathrm{i}}(\hat{\boldsymbol{x}},\hat{t})\to 0$ as $\hat{r}\to\infty$, ψ_0^{i} is the solution of a classical two-dimensional boundary-value problem. Wu (1971*a*) has obtained the solution of this problem for arbitrary, unsteady transverse motion and variable forward speed. For steady-state harmonic oscillations and constant forward speed, his solution for arbitrary airfoil shapes and motions becomes

$$f^{\mathrm{i}}(\hat{\zeta},y,\hat{t}) = \frac{1}{\pi\mathrm{i}}\left[\frac{\hat{\zeta}-c}{\hat{\zeta}+c}\right]^{\frac{1}{2}}\int_{-c}^{c}\left[\frac{c+\hat{\xi}}{c-\hat{\xi}}\right]^{\frac{1}{2}}\frac{\phi(\hat{\xi},y,\hat{t})}{\hat{\xi}-\hat{\zeta}}\,\mathrm{d}\hat{\xi}, \tag{4.6a}$$

$$\phi(\hat{x},y,\hat{t}) = UA_0(y,\hat{t})+\phi_1(\hat{x},y,\hat{t}), \tag{4.6b}$$

$$\phi_1(\hat{x},y,\hat{t}) = -\frac{\mathrm{D}}{\mathrm{D}\hat{t}}\int_{-c}^{x} W_0^{\mathrm{i}}(\hat{\xi},y,\hat{t})\,\mathrm{d}\hat{\xi} \quad (|\hat{x}| \leqslant c(y)), \tag{4.6c}$$

$$UA_0(y,\hat{t}) = \tfrac{1}{2}Ua_0(y,\hat{t})-\frac{1}{\pi}\int_{-c}^{c}\frac{\mathrm{d}\hat{\xi}}{(c^2-\hat{\xi}^2)^{\frac{1}{2}}}\phi_1(\hat{\xi},y,\hat{t}), \tag{4.6d}$$

$$a_0(y,\hat{t}) = b_1(y,\hat{t})-[b_0(y,\hat{t})+b_1(y,\hat{t})]\,C(k), \tag{4.6e}$$

$$b_n(y,\hat{t}) = \frac{2}{\pi}\int_0^{\pi} W_0^{\mathrm{i}}(\hat{x},y,\hat{t})\cos n\theta\,\mathrm{d}\theta, \quad n = 0,1,2,\ldots, \tag{4.6f}$$

$$W_0^{\mathrm{i}}(\hat{x},y,\hat{t}) = \frac{\mathrm{D}}{\mathrm{D}\hat{t}}\hat{h}(\hat{x},y,\hat{t}) = \tfrac{1}{2}b_0(y,\hat{t})+\sum_{n=1}^{\infty} b_n(y,\hat{t})\cos n\theta, \tag{4.6g}$$

where $f^{\mathrm{i}}(\hat{\zeta},y,\hat{t}) = \psi^{\mathrm{i}}(\hat{\boldsymbol{x}},\hat{t})+\mathrm{i}\sigma^{\mathrm{i}}(\hat{\boldsymbol{x}},\hat{t})$ is the complex acceleration potential, $\hat{\zeta} = \hat{x}+\mathrm{i}\hat{z}$, $\hat{x} = c(y)\cos\theta$, i is the spatial complex unit ($\mathrm{ij} \neq -1$), $k = (\omega/U)\,c(y)/A$ is the reduced frequency based on the local semichord. $C(k)$ is Theodorsen's function (Theodorsen 1935):

$$C(k) = \frac{H_1^{(2)}(k)}{H_1^{(2)}(k)+\mathrm{j}H_0^{(2)}(k)}$$

$$= \frac{J_1(J_1+Y_0)+Y_1(Y_1-J_0)}{(J_1+Y_0)^2+(Y_1-J_0)^2}+\mathrm{j}\frac{-(Y_1Y_0+J_1J_0)}{(J_1+Y_0)^2+(Y_1-J_0)^2},$$

where $H_n^{(2)}(z) = J_n(z)-\mathrm{j}Y_n(z)$ is the Hankel function of the second kind of order n, and J_n and Y_n are Bessel functions of, respectively, the first and second kind of order n. Wu's method yields the acceleration potential throughout the flow field, needed for the present analysis. The integrals necessary to calculate the pressure field from (4.6) for most wing displacements of interest are evaluated by Ahmadi (1980).

Using Wu's method, $\tilde{\psi}^{\mathrm{i}}_{2\mathrm{D}}$ is determined:

$$\tilde{\psi}^{\mathrm{i}}_{2\mathrm{D}}(\hat{\mathbf{x}}) = \mathrm{Re}_{\mathrm{i}}\,[\tilde{f}^{\mathrm{i}}_{2\mathrm{D}}(\zeta, y)], \tag{4.7a}$$

$$\tilde{f}^{\mathrm{i}}_{2\mathrm{D}}(\zeta, y) = -\mathrm{i}\,\{B_1(y)\,[-\zeta^2 + (\zeta^2 + c\zeta + \tfrac{1}{2}c^2)\,\lambda] + B_2(y)\,[-\zeta + (\zeta^2 - c^2)^{\frac{1}{2}}] + B_3(y)\,[\lambda - 1]\}, \tag{4.7b}$$

where Re_{i} denotes the real part of a complex quantity with respect to i, k_0 is the reduced frequency based on root semichord c_0/A, and

$$\lambda = \left[\frac{\zeta - c}{\zeta + c}\right]^{\frac{1}{2}}, \tag{4.8a}$$

$$B_1(y) = \frac{1}{2c^2}\,U^2 k^2 \alpha, \tag{4.8b}$$

$$B_2(y) = \frac{1}{c}\,U^2\left[k_0\,k\,\frac{h_0}{c_0} - 2\mathrm{j}k\alpha\right], \tag{4.8c}$$

$$B_3(y) = U^2\left\{-\tfrac{1}{4}(k^2 - 2\mathrm{j}k)\,\alpha - \left[\mathrm{j}k_0\,\frac{h_0}{c_0} + (1 + \tfrac{1}{2}\mathrm{j}k)\,\alpha\right] C(k)\right\}. \tag{4.8d}$$

4.1. *Eigensolutions of the inner acceleration potential*

The eigensolutions of the inner solution satisfy the homogeneous boundary-value problem (4.5*a–e*). We consider two cases. First, we assume $\psi^{\mathrm{i}}(\hat{\mathbf{x}}, \hat{t}) \rightarrow 0$ as $\hat{r} \rightarrow \infty$. The solution to this problem is obtained from Wu's method (for details see Ahmadi 1980), and is the solution of the Sears problem (Sears 1941): the interaction of a convecting sinusoidal gust of constant amplitude with a flat-plate airfoil, for which $\partial\psi^{\mathrm{i}}/\partial\hat{z} = 0$ at the airfoil and $\psi^{\mathrm{i}} \rightarrow \infty$ as $\hat{r} \rightarrow \infty$. Thus

$$\tilde{\psi}^{\mathrm{i}}_{\mathrm{Sears}}(\hat{\mathbf{x}}) = \mathrm{Re}_{\mathrm{i}}\,[\tilde{f}^{\mathrm{i}}_{\mathrm{Sears}}(\zeta, y)], \tag{4.9a}$$

where

$$\tilde{f}^{\mathrm{i}}_{\mathrm{Sears}}(\zeta, y) = -\mathrm{i}U\tilde{W}_{\mathrm{g}}(y)\,S(k)\,[\lambda - 1], \tag{4.9b}$$

$\tilde{W}_{\mathrm{g}}(y)$ is the still-unknown complex amplitude of the sinusoidal gust and $S(k)$ is the Sears function (Sears 1941)

$$S(k) = \mathrm{j}J_1(k) + [J_0(k) - \mathrm{j}J_1(k)]\,C(k).$$

In addition to $\psi^{\mathrm{i}}_{\mathrm{Sears}}$, there is an infinite number of eigensolutions that satisfy (4.5*a–e*) but do not vanish at infinity. These are found by inspection:

$$\tilde{\Psi}^{\mathrm{i}}_1(\hat{\mathbf{x}}) = \tilde{\psi}^{\mathrm{i}}_{2\mathrm{D,\,heave}}(\hat{\mathbf{x}}) + \left(\frac{\omega}{A}\right)^2 \hat{z}, \tag{4.10}$$

$$\tilde{\Psi}^{\mathrm{i}}_2(\hat{\mathbf{x}}) = \tilde{\psi}^{\mathrm{i}}_{2\mathrm{D,\,pitch}}(\hat{\mathbf{x}}) + \left(\frac{\omega}{A}\right)^2 \hat{x}\hat{z} - 2\mathrm{j}\,\frac{\omega U}{A}\,\hat{z}, \tag{4.11}$$

$$\vdots$$

Ψ^{i}_1 and Ψ^{i}_2 consist respectively of the pressure field of an airfoil in heave and pitch of unit amplitude, together with the pressure field necessary to cancel out the resulting vertical acceleration at the airfoil, so that $\partial\psi^{\mathrm{i}}/\partial\hat{z} = 0$. Other eigensolutions involve oscillating airfoils with chordwise bending. We will see that to obtain the leading three-dimensional correction, only $\psi^{\mathrm{i}}_{\mathrm{Sears}}$ is required.

The first three elements of the inner solution are thus given by

$$\tilde{\psi}^{\mathrm{i}}_0(\hat{\mathbf{x}}) = \tilde{\psi}^{\mathrm{i}}_{2\mathrm{D}}(\hat{\mathbf{x}}) + F_0(y)\,\tilde{\psi}^{\mathrm{i}}_{0,\,\mathrm{Sears}} + f_0(y)\,\tilde{\Psi}^{\mathrm{i}}_1(\hat{\mathbf{x}}) + g_0(y)\,\tilde{\Psi}^{\mathrm{i}}_2(\hat{\mathbf{x}}) + \ldots, \tag{4.12}$$

$$\tilde{\psi}^{\mathrm{i}}_k(\hat{\mathbf{x}}) = F_k(y)\,\tilde{\psi}^{\mathrm{i}}_{0,\,\mathrm{Sears}} + f_k(y)\,\tilde{\Psi}^{\mathrm{i}}_1(\hat{\mathbf{x}}) + g_k(y)\,\tilde{\Psi}^{\mathrm{i}}_2(\hat{\mathbf{x}}) + \ldots, \quad k = 1, 2. \tag{4.13}$$

The unknown weighting functions $F_n, f_n, g_n, \ldots$ are assumed to be $O(1)$ and

$$\tilde{\psi}^{\mathrm{i}}_{0,\,\mathrm{Sears}}(\hat{\boldsymbol{x}}) = \mathrm{Im}_{\mathrm{i}}\,[\lambda], \tag{4.14}$$

where we have absorbed the multiplicative term $U\tilde{W}_{\mathrm{g}}(y)\,S(k)$ in $\psi^{\mathrm{i}}_{\mathrm{Sears}}$, (4.9*a*, *b*), into $F_n(y)$. Im_{i} denotes the imaginary part of a complex quantity with respect to i.

4.2. *Expansion of inner solution for small reduced frequencies*

In the present model, since the chord is $O(A^{-1})$, the reduced frequency based on the semichord tends to zero as $A \rightarrow \infty$. Hence the inner solution must be expanded for small k. In particular, the asymptotic expansion for small k of Theodorsen's function is

$$C(k) \sim 1 - \nu\,[\tfrac{1}{2}\pi - \mathrm{j}\log(\tfrac{1}{2}\gamma_1\,\nu)]\,A^{-1} - \mathrm{j}\,\nu A^{-1}\log A + O(A^{-2}\log^2 A), \tag{4.15}$$

where $\log\gamma_1 = \gamma = 0.57721\ldots$ is the Euler constant and $\nu(y) = \omega c(y)/U$ is the reduced frequency based on the magnified semichord $c(y)$. The expansion of $\psi^{\mathrm{i}}_{2\mathrm{D}}$ for small k is given by

$$\tilde{\psi}^{\mathrm{i}}_{2\mathrm{D}}(\hat{\boldsymbol{x}}) \sim \tilde{\psi}^{\mathrm{i}}_{2\mathrm{D},1}(\hat{\boldsymbol{x}}) + A^{-1}\log A\,\tilde{\psi}^{\mathrm{i}}_{2\mathrm{D},2}(\hat{\boldsymbol{x}}) + A^{-1}\tilde{\psi}^{\mathrm{i}}_{2\mathrm{D},3}(\hat{\boldsymbol{x}}) + O(A^{-2}\log^2 A), \tag{4.16a}$$

where

$$\tilde{\psi}^{\mathrm{i}}_{2\mathrm{D},1}(\hat{\boldsymbol{x}}) = -U^2\alpha\,\mathrm{Im}_{\mathrm{i}}\,[\lambda], \tag{4.16b}$$

$$\tilde{\psi}^{\mathrm{i}}_{2\mathrm{D},2}(\hat{\boldsymbol{x}}) = \mathrm{j}\nu U^2\alpha\,\mathrm{Im}_{\mathrm{i}}\,[\lambda], \tag{4.16c}$$

$$\tilde{\psi}^{\mathrm{i}}_{2\mathrm{D},3}(\hat{\boldsymbol{x}}) = -\mathrm{j}\nu U^2\left\{\left\{[\log(\tfrac{1}{2}\gamma_1\,\nu) + \tfrac{1}{2}\mathrm{j}\pi]\,\alpha + \frac{h_0}{c}\right\}\mathrm{Im}_{\mathrm{i}}\,[\lambda] + \frac{2\alpha}{c}\,\mathrm{Im}_{\mathrm{i}}\,[-\zeta + (\zeta^2 - c^2)^{\frac{1}{2}}]\right\} \tag{4.16d}$$

are $O(1)$ quantities and λ is defined in (4.8*a*).

The expansion of $\psi^{\mathrm{i}}_{\mathrm{Sears}}$ is not needed, since $\psi^{\mathrm{i}}_{0,\,\mathrm{Sears}}$ is independent of k. The expansions for the other eigensolutions are obtained from the expansion for $\psi^{\mathrm{i}}_{2\mathrm{D}}$:

$$\tilde{\Psi}^{\mathrm{i}}_{1}(\hat{\boldsymbol{x}}) \sim \left(\frac{\omega}{A}\right)^2 \hat{z} - A^{-1}\,\mathrm{j}\omega U\,\mathrm{Im}_{\mathrm{i}}\,[\lambda] + O(A^{-2}\log^2 A), \tag{4.17}$$

$$\begin{aligned}\tilde{\Psi}^{\mathrm{i}}_{2}(\hat{\boldsymbol{x}}) \sim{}& \left(\frac{\omega}{A}\right)^2 \hat{x}\hat{z} - 2\mathrm{j}\frac{\omega U}{A}\hat{z} - U^2\,\mathrm{Im}_{\mathrm{i}}\,[\lambda] + A^{-1}\log A\,\mathrm{j}\nu U^2\,\mathrm{Im}_{\mathrm{i}}\,[\lambda]\\ &- A^{-1}\,\mathrm{j}\nu U^2\left\{[\log(\tfrac{1}{2}\gamma_1\,\nu) + \tfrac{1}{2}\mathrm{j}\pi]\,\mathrm{Im}_{\mathrm{i}}\,[\lambda] + \frac{2}{c}\,\mathrm{Im}_{\mathrm{i}}\,[-\zeta + (\zeta^2 - c^2)^{\frac{1}{2}}]\right\}\\ &+ O(A^{-2}\log^2 A).\end{aligned} \tag{4.18}$$

4.3. *Outer expansion of inner expansion*

The outer expansion of the inner expansion is obtained from the inner solution in the limit of $\hat{r} \rightarrow \infty$. Thus

$$\begin{aligned}\tilde{\psi}^{\mathrm{io}}_{2\mathrm{D},1}(\hat{\boldsymbol{x}}) \sim{}& \frac{1}{2\pi\rho}\left\{\left[-2\pi\rho U^2\frac{c}{A}\alpha\right]\frac{\sin\theta}{r}\right.\\ &\left.+\left[\pi\rho U^2\left(\frac{c}{A}\right)^2\alpha\right]\frac{\sin 2\theta}{r^2} + \left[-\pi\rho U^2\left(\frac{c}{A}\right)^3\alpha\right]\frac{\sin 3\theta}{r^3} + O(A^{-4})\right\},\end{aligned} \tag{4.19a}$$

$$\tilde{\psi}^{\mathrm{io}}_{2\mathrm{D},2}(\hat{\boldsymbol{x}}) \sim \frac{1}{2\pi\rho}\left\{\left[2\pi\rho U^2\,\mathrm{j}\nu\frac{c}{A}\alpha\right]\frac{\sin\theta}{r} + \left[-\pi\rho U^2\,\mathrm{j}\nu\left(\frac{c}{A}\right)^2\alpha\right]\frac{\sin 2\theta}{r^2} + O(A^{-3})\right\}, \tag{4.19b}$$

$$\tilde{\psi}^{1o}_{2D,3}(\hat{x}) \sim \frac{1}{2\pi\rho}\left\{-2\pi\rho U^2 j\nu\left\{[\log(\tfrac{1}{2}\gamma_1\nu)+\tfrac{1}{2}j\pi+1]\alpha+\frac{h_0}{c}\right\}\frac{c}{A}\frac{\sin\theta}{r}\right.$$
$$\left.+\pi\rho U^2 j\nu\left\{[\log(\tfrac{1}{2}\gamma_1\nu)+\tfrac{1}{2}j\pi]\alpha+\frac{h_0}{c}\right\}\left(\frac{c}{A}\right)^2\frac{\sin 2\theta}{r^2}+O(A^{-3})\right\}, \quad (4.19c)$$

$$\tilde{\psi}^{1o}_{0,\text{Sears}}(\hat{x}) \sim \frac{c}{A}\frac{\sin\theta}{r}-\frac{1}{2}\left(\frac{c}{A}\right)^2\frac{\sin 2\theta}{r^2}+\frac{1}{2}\left(\frac{c}{A}\right)^3\frac{\sin 3\theta}{r^3}+O(A^{-4}), \quad (4.20)$$

$$\tilde{\Psi}^{1o}_1(\hat{x}) \sim \frac{\omega^2}{A}r\sin\theta+\frac{1}{2\pi\rho}\left[-2\pi\rho U^2 j\nu\frac{1}{A^2}\frac{\sin\theta}{r}+O(A^{-3})\right], \quad (4.21)$$

$$\tilde{\Psi}^{1o}_2(\hat{x}) \sim \tfrac{1}{2}\omega^2 r^2\sin 2\theta-2j\omega Ur\sin\theta+\frac{1}{2\pi\rho}\left[-2\pi\rho U^2\frac{c}{A}\frac{\sin\theta}{r}+O(A^{-2})\right], \quad (4.22)$$

$$\vdots$$

This completes the inner solution and its outer expansion. The results of James (1975) for ψ^1 and ψ^{1o} lack the eigensolutions, have not been expanded for small k, have an extra factor of A, and include his induced downwash, which is incorrect.

5. Matching

The outer and inner solutions are matched according to the asymptotic matching principle of Van Dyke (1975). The principal results of a step-by-step application of the matching principle are given below, where m and n denote respectively the number of terms of the inner and outer expansions.

$m = n = 1$:

$$f_0(y) = g_0(y) = \ldots = 0,$$
$$\tilde{l}(y) = -2\pi\rho U^2\frac{c}{A}\alpha+2\pi\rho\frac{c}{A}F_0(y). \quad (5.1)$$

$m = 1,\ n = 2$:

At this level, section lift is the same as in (5.1), and

$$\tilde{m}(y) = -\pi\rho U^2\left(\frac{c}{A}\right)^2\alpha+\pi\rho\left(\frac{c}{A}\right)^2F_0(y). \quad (5.2)$$

$m = n = 2$:

$$f_1(y) = g_1(y) = \ldots = 0,$$
$$\tilde{l}(y) = -2\pi\rho U^2\frac{c}{A}\alpha+2\pi\rho\frac{c}{A}F_0(y)+A^{-1}\log A\left[2\pi\rho U^2 j\nu\frac{c}{A}\alpha\right]$$
$$+A^{-1}\left\{-2\pi\rho U^2 j\nu\left\{[\log(\tfrac{1}{2}\gamma_1\nu)+\tfrac{1}{2}j\pi+1]\alpha+\frac{h_0}{c}\right\}\frac{c}{A}+2\pi\rho\frac{c}{A}F_1(y)\right\}; \quad (5.3)$$

the section moment is the same as in (5.2).

$m = 2,\ n = 3$:

Here the section lift is the same as in (5.3), and

$$\tilde{m}(y) = -\pi\rho U^2\left(\frac{c}{A}\right)^2\alpha+\pi\rho\left(\frac{c}{A}\right)^2F_0(y)+A^{-1}\log A\left[\pi\rho U^2 j\nu\left(\frac{c}{A}\right)^2\alpha\right]$$
$$+A^{-1}\left\{-\pi\rho U^2 j\nu\left\{[\log(\tfrac{1}{2}\gamma_1\nu)+\tfrac{1}{2}j\pi]\alpha+\frac{h_0}{c}\right\}\left(\frac{c}{A}\right)^2+\pi\rho\left(\frac{c}{A}\right)^2F_1(y)\right\}, \quad (5.4)$$

$$\tilde{q}(y) = -\pi\rho U^2\left(\frac{c}{A}\right)^3\alpha+\pi\rho\left(\frac{c}{A}\right)^3F_0(y). \quad (5.5)$$

The above results show that to $O(A^{-2})$ the solution for the acceleration potential contains no eigensolutions except possibly $\psi^{\mathrm{i}}_{0,\,\mathrm{Sears}}$. To leading order, the section lift and moment consist of their two-dimensional quasi-steady values. As the matching proceeds to higher orders, sectional loads are refined with two-dimensional unsteady information and possible contributions from $\psi^{\mathrm{i}}_{0,\,\mathrm{Sears}}$. We will see that the latter contain the leading three-dimensional correction.

We now construct the composition solution

$$\psi_{\mathrm{c}} = \psi^{\mathrm{i}} + \psi^{\mathrm{o}} - \psi^{\mathrm{oi}}, \tag{5.6}$$

where $\psi^{\mathrm{oi}} = \psi^{\mathrm{io}}$ is the common solution. To $O(A^{-2})$, ψ^{i} is given by

$$\tilde{\psi}^{\mathrm{i}}(\hat{\boldsymbol{x}}) \sim \tilde{\psi}^{\mathrm{i}}_{\mathrm{2D},\,1}(\hat{\boldsymbol{x}}) + F_0(y)\,\tilde{\psi}^{\mathrm{i}}_{0,\,\mathrm{Sears}}(\hat{\boldsymbol{x}}) + A^{-1}\log A\,\tilde{\psi}^{\mathrm{i}}_{\mathrm{2D},\,2}(\hat{\boldsymbol{x}}) + A^{-1}\,[\tilde{\psi}^{\mathrm{i}}_{\mathrm{2D},\,3}(\hat{\boldsymbol{x}}) + F_1(y)\,\tilde{\psi}^{\mathrm{i}}_{0,\,\mathrm{Sears}}(\hat{\boldsymbol{x}})]. \tag{5.7a}$$

Using (4.16), we may rewrite this as

$$\tilde{\psi}^{\mathrm{i}}(\hat{\boldsymbol{x}}) = \tilde{\psi}^{\mathrm{i}}_{\mathrm{2D}}(\hat{\boldsymbol{x}}) + [F_0(y) + A^{-1}\log A F_1(y)]\,\tilde{\psi}^{\mathrm{i}}_{0,\,\mathrm{Sears}}(\hat{\boldsymbol{x}}) \tag{5.7b}$$

without altering its accuracy. Using (5.7 *b*) will greatly facilitate downwash calculation in §6. Furthermore, to $O(A^{-2})$,

$$\tilde{\psi}^{\mathrm{o}}(\boldsymbol{x}) = \frac{-1}{4\pi\rho}\frac{\partial}{\partial z}\int_{-b}^{b}\frac{\tilde{l}_0(\eta)}{[x^2+(y-\eta)^2+z^2]^{\frac{1}{2}}}\,\mathrm{d}\eta, \tag{5.8}$$

$$\tilde{\psi}^{\mathrm{oi}}(\boldsymbol{x}) = \frac{1}{2\pi\rho}\tilde{l}_0(y)\frac{z}{x^2+z^2}, \tag{5.9}$$

where $\tilde{l}_0(y) = -2\pi\rho U^2(c/A)\,\alpha$ is the two-dimensional quasi-steady section lift. The above solution is not unique, since it contains multiples of the eigensolution $\psi^{\mathrm{i}}_{0,\,\mathrm{Sears}}$, as indicated by the unknown weighting functions $F_n(y)$.

6. Uniqueness

Uniqueness of the solution is achieved by calculating the downwash by integration of the composite pressure field from far upstream to some point on the wing. Downwash at a field point $\boldsymbol{x}$ is given by

$$\tilde{W}(\boldsymbol{x}) = \frac{1}{U}\int_{-\infty}^{x}\frac{\partial}{\partial z}[\tilde{\psi}^{\mathrm{c}}(\xi, y, z)]\,\mathrm{e}^{\mathrm{j}\bar{\omega}(\xi - x)}\,\mathrm{d}\xi, \tag{6.1}$$

where $\bar{\omega} = \omega/U$. For points on the wing, the integration path passes over (or under) the leading edge, where the vertical acceleration of a fluid particle

$$\frac{\partial\psi_{\mathrm{c}}}{\partial z} \sim \frac{\partial\psi^{\mathrm{i}}}{\partial\hat{z}} \sim \mathrm{Re}_{\mathrm{i}}\,[(\zeta + c)^{-\frac{3}{2}}] \quad (\zeta \rightarrow -c(y))$$

has a non-integrable singularity. First we calculate the downwash at the wing due to ψ^{i} (which contains the singularity) and denote it by W^{i}.

6.1. *Calculation of* $W^{\mathrm{i}}(x, y, z, t)$ *at the wing*

We begin by inverting the linearized Euler equation, written for the cross-sectional plane of the wing in complex form,

$$\frac{\partial}{\partial\zeta}f^{\mathrm{i}}(\zeta, y, t) = \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial\zeta}\right)g^{\mathrm{i}}(\zeta, y, t), \tag{6.2}$$

to obtain the complex velocity $g^i(\zeta, y, t) = u^i(\hat{x}, t) - \mathrm{i}w^i(\hat{x}, t)$:

$$\tilde{g}^i(\zeta, y) = \frac{1}{U}\int_{-\infty}^{\hat{\zeta}} \frac{\partial}{\partial \zeta_1}[\tilde{f}^i(\zeta_1, y)]\, \mathrm{e}^{\mathrm{j}\bar{\omega}(\hat{\zeta}_1 - \hat{\zeta})}\, \mathrm{d}\zeta_1. \tag{6.3}$$

We resolve the difficulty at the leading edge by considering the general case $\hat{z} \neq 0$, integrating by parts, and thereafter letting $\hat{z} \rightarrow 0\pm$. Since downwash is an even function of $\hat{z}$, it suffices to consider $\hat{z} \rightarrow 0+$ only. The downwash at the wing thus becomes

$$\tilde{W}^i(x, y, 0+) = \frac{-1}{U}\,\mathrm{Im}_i\,[\tilde{f}^i(x + \mathrm{i}0+, y)]$$
$$+\frac{\mathrm{j}\bar{\omega}}{UA}\int_{-\infty}^{\hat{x}} \mathrm{Im}_i\,[\tilde{f}^i(\xi + \mathrm{i}0+, y)]\, \mathrm{e}^{\mathrm{j}\bar{\omega}(\hat{\xi} - \hat{x})A^{-1}}\, \mathrm{d}\xi \quad (|\hat{x}| \leqslant c(y),\ |y| \leqslant b), \tag{6.4}$$

where we have made use of the fact that $f^i \rightarrow 0$ as $\zeta \rightarrow \infty$ (see below).

To $O(A^{-2})$, f^i consists of f^i_{2D} and $f^i_{0,\,\mathrm{Sears}}$, where the latter is the complex form of $\psi^i_{0,\,\mathrm{Sears}}$ obtained from (4.9*b*) after removing the factor $U\tilde{W}_g(y)\, S(k)$. Since both f^i_{2D} and $f^i_{0,\,\mathrm{Sears}}$ are two-dimensional, they vanish as $\hat{r} \rightarrow \infty$. It can be shown that substituting f^i_{2D} into (6.4) yields the prescribed downwash at the wing, as expected. Similarly, the downwash at the wing due to $f^i_{0,\,\mathrm{Sears}}$, say $W^i_{0,\,\mathrm{Sears}}$, is found to be

$$\tilde{W}^i_{0,\,\mathrm{Sears}}(\hat{x}, y, 0+) = \frac{\mathrm{j}\pi}{2U} k\, \mathrm{e}^{-\mathrm{j}\bar{\omega}x}\,[H_1^{(2)}(k) + \mathrm{j}H_0^{(2)}(k)]$$
$$\sim \frac{-1}{U}\,\mathrm{e}^{-\mathrm{j}\bar{\omega}x}[1 + O(A^{-1}\log A)]. \tag{6.5}$$

Substituting the above results for W_i for points on the wing in (6.1) and setting the computed downwash equal to the prescribed value W^i_0, we obtain

$$\tilde{W}^i_0(\hat{x}, y) = \tilde{W}^i_0(\hat{x}, y) - \frac{1}{U}\,\mathrm{e}^{-\mathrm{j}\bar{\omega}x}\,[F_0(y) + A^{-1}F_1(y)]$$
$$+ \lim_{z \rightarrow 0+}\,[\tilde{W}^{\mathrm{o}}(x, y, z) - \tilde{W}^{\mathrm{oi}}(x, y, z)]\,\mathrm{e}^{-\mathrm{j}\bar{\omega}x} \quad (|x| \leqslant c(y)/A,\quad |y| \leqslant b), \tag{6.6}$$

where

$$\mathrm{e}^{-\mathrm{j}\bar{\omega}x}\,\tilde{W}^{\mathrm{o}}(x, y, z) = \frac{1}{U}\int_{-\infty}^{x} \frac{\partial}{\partial z}[\tilde{\psi}^{\mathrm{o}}(\xi, y, z)]\,\mathrm{e}^{\mathrm{j}\bar{\omega}(\xi - x)}\,\mathrm{d}\xi, \tag{6.7}$$

$$\mathrm{e}^{-\mathrm{j}\bar{\omega}x}\,\tilde{W}^{\mathrm{oi}}(x, y, z) = \frac{1}{U}\int_{-\infty}^{x} \frac{\partial}{\partial z}[\tilde{\psi}^{\mathrm{oi}}(\xi, y, z)]\,\mathrm{e}^{\mathrm{j}\bar{\omega}(\xi - x)}\,\mathrm{d}\xi \tag{6.8}$$

are respectively the downwash velocities due to the outer and common solutions. In (6.6) the downwash velocity due to ψ^i_{2D} identically cancels with the prescribed downwash at the wing, since all of the wing boundary condition was used to determine the lowest-order inner solution ψ^i_0.

We now consider the balance of the two remaining terms in (6.6). After cancelling the common sinusoidal dependence on x, we conclude that, since the first term is independent of x, the second term must be also. Hence we need to evaluate the second term for only one value of x: for convenience, $x = 0$. In the limit of $z \rightarrow 0+$, $\tilde{W}^{\mathrm{o}}(0, y, z)$ is the downwash due to ψ^{o} near the loaded line and $\tilde{W}^{\mathrm{oi}}(0, y, z)$ is the downwash due to ψ^{oi} near the two-dimensional dipole of strength $\tilde{l}_0(y)$. All that remains to determine F_0, F_1 and F_2 is to determine $\lim_{z \rightarrow 0+}[\tilde{W}^{\mathrm{o}}(0, y, z) - \tilde{W}^{\mathrm{oi}}(0, y, z)]$.

6.2. *Calculation of $W^o(0, y, z, t)$ as $z \to 0+$*

$\tilde{W}^o(0, y, z)$ is obtained by substituting (5.8) into (6.7):

$$\tilde{W}^o(0, y, z) = \frac{1}{4\pi\rho U} \int_{-b}^{b} \mathrm{d}\eta \, \tilde{l}_0(\eta) \left\{ -\frac{\partial^2}{\partial z^2} \int_{-\infty}^{0} \frac{\mathrm{e}^{\mathrm{j}\bar{\omega}\lambda}}{[\lambda^2 + y_0^2 + z^2]^{\frac{3}{2}}} \mathrm{d}\lambda \right\}. \quad (6.9)$$

The expression in the braces is the kernel function of unsteady lifting-surface theory for incompressible flow, arbitrary z, and $x_0 = 0$, i.e. $K(x_0 = 0, y_0, z)$ in the standard notation ($x_0 = x - \zeta$, $y_0 = y - \eta$). $K(x_0, y_0, z)$ can be evaluated in terms of special functions (for arbitrary z, see Widnall 1964). For $x_0 = 0$,

$$K(0, y_0, z) = \frac{\bar{\omega}}{r_1} \{K_1(\bar{\omega} r_1) + \tfrac{1}{2}\mathrm{j}\pi [I_1(\bar{\omega} r_1) - L_1(\bar{\omega} r_1)] - \mathrm{j}\}$$
$$- \frac{\bar{\omega}^2 z^2}{r_1^2} \left\{ K_2(\bar{\omega} r_1) - \tfrac{1}{2}\mathrm{j}\pi [I_2(\bar{\omega} r_1) - L_2(\bar{\omega} r_1)] + \tfrac{1}{3}\mathrm{j}\, \bar{\omega} r_1 - \frac{\mathrm{j}}{\bar{\omega} r_1} \right\}, \quad (6.10)$$

where $r_1 = (y_0^2 + z^2)^{\frac{1}{2}}$, and I_n, K_n and L_n are respectively modified Bessel functions of the first and second kind of order n and the modified Struve function of order n. For computational purposes, it is often inconvenient to evaluate I_n and L_n separately, but their difference are neatly expressible by formulas like

$$I_1(\mu) - L_1(\mu) = \frac{2}{\pi} \left[1 - \int_0^{\frac{1}{2}\pi} \mathrm{e}^{-\mu \cos\theta} \cos\theta \, \mathrm{d}\theta \right].$$

$I_1 - L_1$ can also be evaluated from a closed-form approximation given by Watkins, Woolston & Cunningham (1959).

To understand the nature of the singularities of $K(0, y_0, z)$ and $\tilde{W}^o(0, y, z)$ as $z \to 0+$, we adopt the vortex viewpoint where the outer solution is a harmonically oscillating concentrated vortex accompanied by a wake of trailing and shed vorticity. The contribution of the trailing vorticity to downwash at the loaded line is finite and involves the classical second-order singularity of wing theory (in K) in the spanwise direction (Watkins *et al.* 1959). The contribution of the shed vorticity involves a logarithmic singularity, an idea familiar from lifting-surface theory.

Formally, we substitute (6.10) into (6.9) and derive an expansion for the integral in the limit of $z \to 0+$ (see Ahmadi 1980):

$$\tilde{W}^o(0, y, z) \sim \frac{1}{4\pi\rho U} \,\,⨎_{-b}^{b} \frac{\tilde{l}_0(\eta)}{(y-\eta)^2} \Sigma(\bar{\omega}\,|y-\eta|)\, \mathrm{d}\eta$$
$$- \frac{\mathrm{j}\bar{\omega}}{4\pi\rho U} \int_{-b}^{b} \frac{\tilde{l}_0(\eta) - \tilde{l}_0(y)}{|y-\eta|} \mathrm{d}\eta - \frac{\mathrm{j}\bar{\omega}}{4\pi\rho U} \tilde{l}_0(y) \left\{ -2 - 2\log\frac{z}{b} + \log 4 \left[1 - \left(\frac{y}{b}\right)^2 \right] \right\}$$
$$+ O(z^2 \log z) \quad (|y| \leqslant b), \quad (6.11a)$$

where ⨎ denotes the principal value of the integral in the sense of Hadamard (see Mangler 1951) and

$$\Sigma(\mu) = \mu\{K_1(\mu) + \tfrac{1}{2}\mathrm{j}\pi [I_1[\mu) - L_1(\mu)]\}. \quad (6.11b)$$

The real and imaginary parts of Σ are shown in figure 2.

The unsteady induced downwash of James (1975) is closely related to $\tilde{W}^o(0, y, z)$.

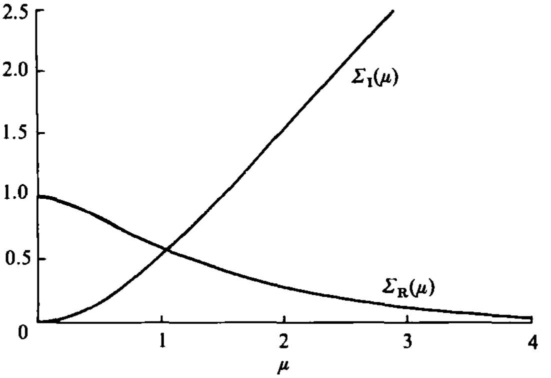

FIGURE 2. The real and imaginary parts of the kernel function of unsteady lifting-line theory $\Sigma(\mu) = \Sigma_R(\mu) + j\Sigma_I(\mu)$.

In the present notation, his result is given by

$$\tilde{W}_I(0, y, 0) \sim \frac{1}{4\pi\rho U}\int_{-b}^{b} \frac{\tilde{l}(\eta)}{(y-\eta)^2}\bar{\Sigma}(\bar{\omega}\,|y-\eta|)\,d\eta \tag{6.12a}$$

where

$$\bar{\Sigma}(\mu) = \Sigma(\mu) - j\mu \tag{6.12b}$$

is essentially the first term (the first pair of braces) of $K(0, y_0, z)$, (6.10). However, his integral does not have a finite value because it contains a non-removable logarithmic singularity (arising from the $-j\mu$ term in Σ) and hence is infinite. This singularity is removed in the present formulation.

6.3. *Calculation of $W^{oi}(x, y, z, t)$ as $z \to 0+$*

From the vortex viewpoint, $\tilde{W}^{oi}(0, y, z)$ is the self-induced downwash at a harmonically oscillating two-dimensional vortex accompanied by a wake of shed vorticity. It too contains a logarithnmic singularity due to the presence of the shed vorticity. $\tilde{W}^{oi}(0, y, z)$ is obtained from (5.9) and (6.8):

$$\tilde{W}^{oi}(0, y, z) = \frac{1}{2\pi\rho U}\tilde{l}_0(y)\left\{\frac{\partial}{\partial z}\int_{-\infty}^{0} \frac{z}{\lambda^2 + z^2}e^{j\bar{\omega}\lambda}\,d\lambda\right\}. \tag{6.13}$$

The expression in the braces is the kernel function of unsteady airfoil theory for incompressible flow, arbitrary z, and $x_0 = 0$, i.e $K_{2D}(x_0 = 0, z)$. $K_{2D}(x_0, z)$ can be evaluated in terms of special functions (see Ahmadi 1980):

$$K_{2D}(x_0, z) = \frac{-x_0}{x_0^2 + z^2} + \tfrac{1}{2}j\bar{\omega}\left\{e^{-q_1}\,\mathrm{Ei}\,(q_1) + e^{-q_2}\left[\mathrm{Ei}\,(q_2) + \pi j\left(1 + \frac{x_0}{|x_0|}\right)\right]\right\}, \tag{6.14}$$

where $q_1 = \bar{\omega}(-z + jx_0)$, $q_2 = \bar{\omega}(z + jx_0)$,

$$\frac{x_0}{|x_0|} = \begin{cases} 1 & (x_0 > 0), \\ 0 & (x_0 = 0), \\ -1 & (x_0 < 0), \end{cases}$$

and $\mathrm{Ei}\,(\zeta)$ is the complex exponential integral

$$\mathrm{Ei}\,(\zeta) = \int_{-\infty}^{\zeta} \frac{e^t}{t}\,dt,$$

defined with a branch cut along the positive real axis. Setting $x_0 = 0$ and expanding for $z \to 0+$ (the necessary expansions are found in Erdélyi 1953; Gröbner & Hofreiter 1961), we obtain

$$\tilde{W}^{oi}(0,y,z) \sim \frac{1}{2\pi\rho U} \mathrm{j}\bar{\omega}\tilde{l}_0(y)\left[\gamma + \tfrac{1}{2}\mathrm{j}\pi + \log(\bar{\omega}z)\right] + O(z). \tag{6.15}$$

We notice that the logarithmic term in z in $\tilde{W}^{oi}(0,y,z)$ is identically equal to that in $\tilde{W}^{o}(0, y, z)$, as expected.

It follows from (6.11*a*) and (6.15) that

$$\begin{aligned}\lim_{z\to 0+} &[\tilde{W}^{o}(0,y,z) - \tilde{W}^{oi}(0,y,z)] \\ &= \frac{1}{4\pi\rho U}\left\{ \mathop{\not{\int}}_{-b}^{b} \frac{\tilde{l}_0(\eta)}{(y-\eta)^2} \Sigma(\bar{\omega}\,|y-\eta|)\,\mathrm{d}\eta - \mathrm{j}\bar{\omega}\int_{-b}^{b} \frac{\tilde{l}_0(\eta) - \tilde{l}_0(y)}{|y-\eta|}\,\mathrm{d}\eta \right. \\ &\left. \quad + 2\mathrm{j}\bar{\omega}\,\tilde{l}_0(y)\left\{1 - \gamma - \tfrac{1}{2}\mathrm{j}\pi - \log\mu_0 - \tfrac{1}{2}\log 4\left[1 - \left(\frac{y}{b}\right)^2\right]\right\}\right\} = O(A^{-1}),\end{aligned} \tag{6.16}$$

where $\mu_0 = \bar{\omega}b$ is the reduced frequency based on semispan. Using (6.16) in (6.6), and recalling that $F_n(y) = O(1)$, we find

$$\begin{aligned} F_0(y) &= 0, \\ F_1(y) &= UA \lim_{z\to 0+} [\tilde{W}^{o}(0, y, z) - \tilde{W}^{oi}(0,y,z)] = O(1). \end{aligned}$$

This completes the analysis to $O(A^{-2})$. The F_1 term in the solution represents the leading three-dimensional correction, which is of relative order $O(A^{-1})$.

In summary, the pressure field is given by (5.6)–(5.9), and the section lift and moment are given respectively by (5.3) and (5.4). The matching results show that, in the MAE analysis, the pressure field and airloads first take on their two-dimensional quasi-steady values. As the analysis is carried out to higher orders, they are increasingly refined with two- and three-dimensional unsteady information. It is seen that, for low reduced frequencies, unsteady three-dimensional effects come in at the same order as steady three-dimensional effects. The present analysis is asymptotic for large aspect ratio and small reduced frequency; the results are expected ultimately to diverge with increasing reduced frequency and decreasing aspect ratio.

7. Unsteady induced downwash

Now, we identify unsteady induced downwash, because it contains all of the three-dimensional unsteady effects. We return to (6.6), which equates the downwash, computed from integration of the composite pressure field, with the prescribed downwash at the wing. F_0 and F_1 are given by (6.17). The first term on the right-hand side of (6.6) is the downwash at the wing due to the two-dimensional solution $\psi^{i}_{2\mathrm{D}}$, which equals the prescribed downwash and is cancelled by the left-hand side. The second term is the downwash at the wing due to $\psi^{i}_{0,\,\mathrm{Sears}}$, the local modification of the inner solution arising in response to the induced downwash. The third term on the right-hand side is the downwash at the wing due to the outer solution minus the common solution.

Therefore the last term on the right-hand side of (6.6) is the unsteady induced downwash itself. This term, apart from the common sinusoidal dependence on x, is independent of x. Hence x can be set equal to any constant value on the wing; we

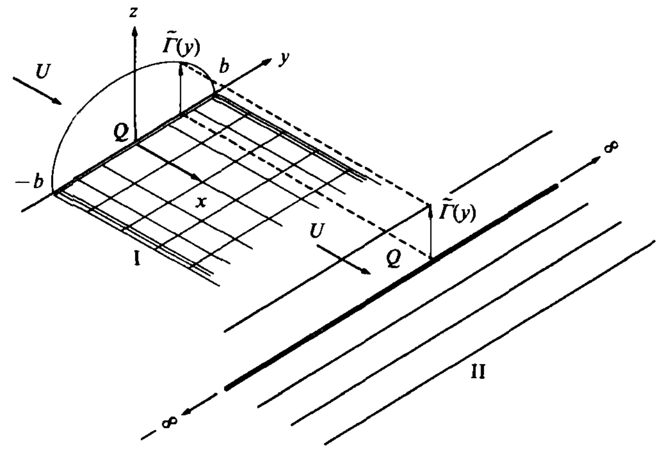

FIGURE 3. Interpretation of unsteady induced downwash (after Van Holten 1976).

choose $x = 0$. Therefore the upper limits of the integrals in (6.7) and (6.8) become zero. The unsteady induced downwash W_I is then given by

$$W_I(x, y, t) = \tilde{W}_g(y)\, e^{-j\bar{\omega}x}\, e^{j\omega t} \quad \left(|x| \leqslant \frac{c(y)}{A}, \quad |y| \leqslant b\right), \tag{7.1a}$$

where

$$\begin{aligned} \tilde{W}_g(y) &= \lim_{z\to 0+} [\tilde{W}^{o}(0, y, z) - \tilde{W}^{oi}(0, y, z)] \\ &= O(A^{-1}) \end{aligned} \tag{7.1b}$$

is given by (6.16). Since for points on the wing $x = O(A^{-1})$, to leading order, (7.1a) reduces to

$$W_I(x, y, t) = \tilde{W}_g(y)\, e^{j\omega t} \quad \left(|x| \leqslant \frac{c(y)}{A}, \quad |y| \leqslant b\right), \tag{7.1c}$$

which is constant across the chord, as in the steady case.

7.1. *Interpretation of induced downwash*

According to (7.1b, c), to leading order, unsteady induced downwash at a spanwise station is made up of the downwash due to vortex system I, $\tilde{W}^{o}(0, y, 0+)$, and that due to vortex system II, $\tilde{W}^{oi}(0, y, 0+)$, as shown in figure 3. Vortex system I is the outer solution: a harmonically oscillating loaded line of strength $\tilde{\Gamma}(y)$, accompanied by a wake of trailing and shed vorticity. Vortex system II is the common solution: a harmonically oscillating two-dimensional vortex of strength $\tilde{\Gamma}(y)$, accompanied by a wake of shed vorticity. The downwash due to both vortex systems is logarithmically infinite, but their difference, which is the unsteady induced downwash, is finite.

This interpretation of induced downwash was first given by Van Holten (1976). However, as cited earlier, he incorrectly assumed constant induced downwash across the chord for arbitrary reduced frequency.

This also resolves the main error in the unsteady lifting-line theory of James (1975). As pointed out earlier, his induced downwash is essentially $\tilde{W}^{o}(0, y, 0+)$, and likewise is logarithmically infinite. In the present theory, induced downwash is inferred from the completed solution, *a posteriori*. James, on the other hand, intuitively defined W_I on the basis of the outer solution alone and used it as the means for connecting the inner and outer solution.

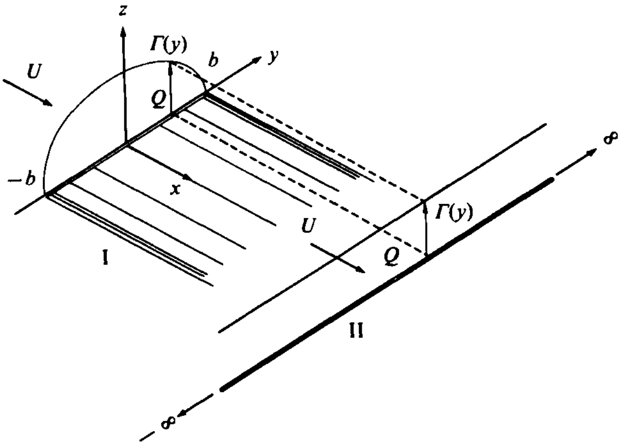

FIGURE 4. Interpretation of steady induced downwash (after Van Holten 1976).

The steady induced downwash is similar to the unsteady one except that the shed vorticity is absent from both vortex systems I and II, as shown in figure 4. Since $\tilde{W}^{\mathrm{oi}}(0, y, 0+)$ is entirely due to unsteady effects, it vanishes in the steady-flow limit. Hence steady induced downwash is entirely due to the trailing vorticity of system I (Prandtl's result).

We can express the results of the present theory directly in terms of unsteady induced downwash by expressing $F_1(y)$ in terms of W_{I}. Substituting (7.1*b, c*) into (6.17), we obtain

$$F_1(y) = U^2 A \tilde{W}_{\mathrm{I}}(y)/U, \tag{7.2}$$

where $\tilde{W}_{\mathrm{I}}(y)/U$ may be thought of as the unsteady induced angle of attack, which varies harmonically with time.

7.2. *An improvement*

The present asymptotic analysis involves a number of exact solutions from unsteady airfoil theory which have been expanded for small reduced frequency or large aspect ratio, with only the first few terms retained to the order of the asymptotic analysis. However, we propose replacing these asymptotic expansions with their exact functional forms, which are valid to at least the order of the corresponding asymptotic expression. This is expected to improve the numerical accuracy of the results over an expanded frequency range.

For $k \to 0$ the induced downwash is constant across the chord. However, to increase the accuracy of the unsteady induced downwash for finite k, we will restore the sinusoidal dependence on x and replace the quasi-steady strip-theory section lift $\tilde{l}_0(y)$ with its exact unsteady counterpart $\tilde{l}_{2\mathrm{D}}(y)$. The improved unsteady induced downwash is then given by

$$W_{\mathrm{I}}(x, y, t) = \tilde{W}_{\mathrm{g}}(y)\, \mathrm{e}^{-\mathrm{j}\bar{\omega}x}\, \mathrm{e}^{\mathrm{j}\omega t} \quad \left(|x| \leqslant \frac{c}{A}, \quad |y| \leqslant b\right), \tag{7.3}$$

where $\tilde{W}_{\mathrm{g}}(y)$ is given by (6.16) with $\tilde{l}_0(y)$ replaced by $\tilde{l}_{2\mathrm{D}}(y)$.

It is seen that the three-dimensional effects at each wing section are manifested as a sinusoidal gust convecting with the free stream, whose complex amplitude $\tilde{W}_{\mathrm{g}}(y)$ varies across the span in a manner determined by wing shapes and motions. We refer to W_{I} as the induced gust. The three-dimensional correction to the basic two-dimensional inner solution is then the pressure field due to the interaction of this induced gust with the wing sections:

$$\tilde{\psi}^{\mathrm{i}}(\hat{\boldsymbol{x}}) = \tilde{\psi}^{\mathrm{i}}_{2\mathrm{D}}(\hat{\boldsymbol{x}}) + \tilde{\psi}^{\mathrm{i}}_{\mathrm{Sears}}(\hat{\boldsymbol{x}}). \tag{7.4}$$

Consequently, the improved three-dimensional section lift and moment consist of the two-dimensional unsteady quantities plus those due to $\tilde{\psi}^{\mathrm{i}}_{\mathrm{Sears}}$, i.e.

$$\tilde{l}(y) = \tilde{l}_{2\mathrm{D}}(y) + \tilde{l}_{\mathrm{Sears}}(y), \quad \tilde{m}(y) = \tilde{m}_{2\mathrm{D}}(y) + \tilde{m}_{\mathrm{Sears}}(y). \tag{7.5}$$

The improved form of the outer solution and the common solution are obtained respectively from (5.8) and (5.9) after replacing $\tilde{l}_0(y)$ by $\tilde{l}_{2\mathrm{D}}(y)$.

$\tilde{l}_{2\mathrm{D}}$ and $\tilde{m}_{2\mathrm{D}}$ are determined by integrating the wing pressure distribution $\Delta\tilde{p}_{2\mathrm{D}}(x, y) = \rho[\tilde{\psi}^{\mathrm{i}}_{2\mathrm{D}}(\hat{x}, y, 0+) - \tilde{\psi}^{\mathrm{i}}_{2\mathrm{D}}(\hat{x}, y, 0-)]$, obtained from (4.7$a$, b). Thus

$$\tilde{l}_{2\mathrm{D}}(y) = -\pi\rho U^2 \frac{c}{A}\left\{\mathrm{j}k\alpha - k_0 k\frac{h_0}{c_0} + \left[(2+\mathrm{j}k)\,\alpha + 2\mathrm{j}k_0\frac{h_0}{c_0}\right]C(k)\right\}, \tag{7.6}$$

$$\tilde{m}_{2\mathrm{D}}(y) = \frac{\pi}{2}\rho U^2\left(\frac{c}{A}\right)^2\left\{(\mathrm{j}k - \tfrac{1}{4}k^2)\,\alpha - \left[(2+\mathrm{j}k)\,\alpha + 2\mathrm{j}k_0\frac{h_0}{c_0}\right]C(k)\right\}, \tag{7.7}$$

which are the familiar results for an airfoil in combined pitch and heave. Similarly, $\tilde{l}_{\mathrm{Sears}}$ and $\tilde{m}_{\mathrm{Sears}}$ are obtained from (4.9a, b):

$$\tilde{l}_{\mathrm{Sears}}(y) = 2\pi\rho U^2 \frac{c}{A}\frac{\tilde{W}_{\mathrm{g}}(y)}{U} S(k), \tag{7.8}$$

$$\tilde{m}_{\mathrm{Sears}}(y) = \pi\rho U^2\left(\frac{c}{A}\right)^2 \frac{\tilde{W}_{\mathrm{g}}(y)}{U} S(k). \tag{7.9}$$

In the limit of steady flow, the present theory reproduces the results of the steady lifting-line theory of Van Dyke (1963). In the remainder of this work, we will use the above extended version of the present theory.

8. Numerical examples and region of validity

Numerical methods have been developed to calculate unsteady induced downwash and sectional and total lift and moment coefficients for oscillating rigid wings. A few examples are presented below. Numerous examples showing the influence of reduced frequency, aspect ratio, wing planform shape and mode of oscillation on the aerodynamics of the wing are given in Ahmadi (1980). They show that, within the region of validity of the theory, with increasing reduced frequency and/or aspect ratio, the three-dimensional results approach their strip-theory counterparts, as expected.

The following examples are for a rigid elliptic wing oscillating in pitch and heave and the extended version of the present theory. The accuracy of the numerical results is three decimal places or better. Without loss of generality, semispan length is taken to be unity.

Figure 5 shows the spanwise distribution of amplitude and phase of the complex amplitude of unsteady induced downwash $\tilde{W}^*_{\mathrm{gP}}$ for an elliptic wing in pitch for several values of reduced frequency k_0. The wing motion is described by $h(x, y, t) = \xi_1 x\, \mathrm{e}^{\mathrm{j}\omega t}$, $\tilde{W}^*_{\mathrm{gP}}(y^*) \equiv \tilde{W}_{\mathrm{g}}(y)/U\xi_1$; and $y^* = y/b$. The station closest to the tip where calculations have been carried out is $y^* = 0.999$. We see that, for spanwise stations not very close to the tip, the amplitude of induced downwash diminishes with increasing reduced

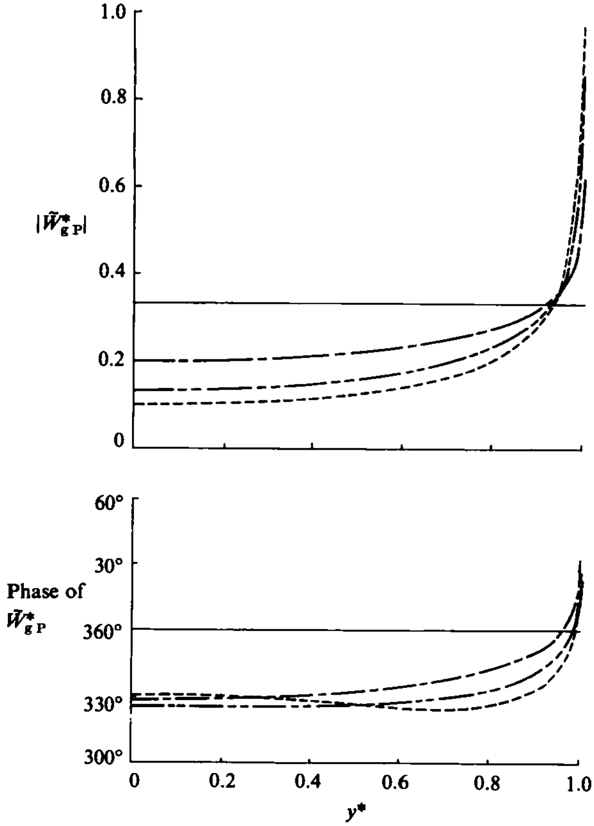

FIGURE 5. Amplitude and phase of $\tilde{W}^*_{gP}$ for an elliptic wing in pitch ($A = 6$): ——, $k_0 = 0$; —–—, 0.1; —–-–—, 0.2; ----, 0.3.

frequency, as expected. In a small neighbourhood of the tip, however, with increasing k_0, the amplitude of induced downwash becomes more intense (becoming possibly infinite at the tip, $y^* = 1$). The latter is due to the increase in the strength of local wake vorticity near the blunt tip which grows stronger with increasing k_0.

Figure 6 shows the total lift and moment coefficients for an elliptic wing in heave, as complex vector diagrams for a range of values of k_0. Figure 7 shows the same results for an elliptic wing in pitch about the midchord line. The heaving motion is described by $h(x, y, t) = \frac{1}{2}(c_0/A)\,\xi_0\,e^{j\omega t}$ and

$$\tilde{C}_{LH} = \int_{-b}^{b} \frac{\tilde{l}(y)\,dy}{\frac{1}{2}\rho U^2 S_a(\frac{1}{2}jk_0\,\xi_0)}, \tag{8.1}$$

$$\tilde{C}_{MH} = \int_{-b}^{b} \frac{\tilde{m}(y)\,dy}{\frac{1}{2}\rho U^2 S_a(2c_0/A)\,\frac{1}{2}jk_0\,\xi_0}, \tag{8.2}$$

$$\tilde{C}_{LP} = \int_{-b}^{b} \frac{\tilde{l}(y)\,dy}{\frac{1}{2}\rho U^2 S_a\,\xi_1}, \tag{8.3}$$

$$\tilde{C}_{MP} = \int_{-b}^{b} \frac{\tilde{m}(y)\,dy}{\frac{1}{2}\rho U^2 S_a(2c_0/A)\,\xi_1}, \tag{8.4}$$

where $(\ )_H$ and $(\ )_P$ denote heave and pitch respectively, and $\frac{1}{2}jk_0\,\xi_0$ is the negative of the angle due to the heaving motion. Shown are the unsteady strip-theory results,

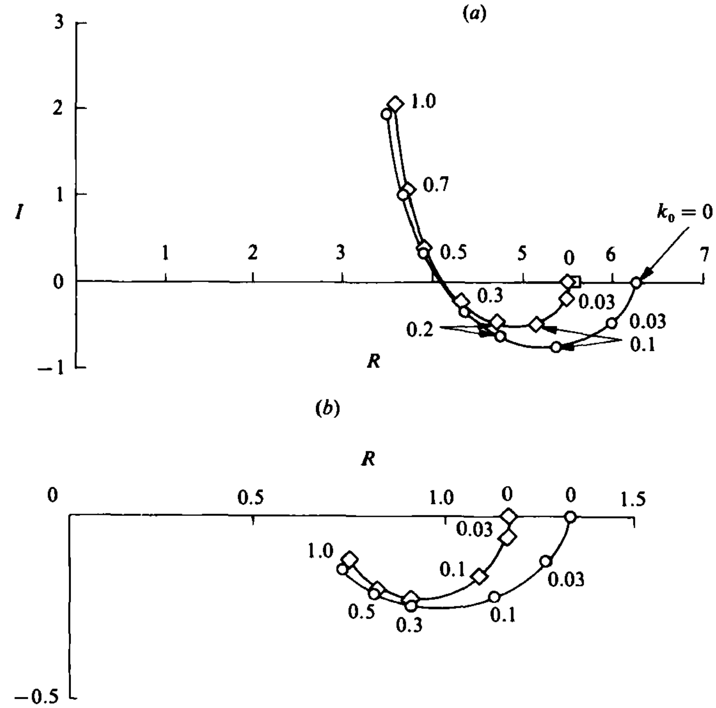

FIGURE 6. Complex vector diagram of $-\tilde{C}_{L_H}$ and $-\tilde{C}_{M_H}$ as functions of k_0 for an elliptic wing in heave ($A = 16$); R and I respectively denote real and imaginary parts of the coefficients; —○—, strip theory; —◇—, unsteady lifting-line theory; □, steady lifting-surface theory.

results of the present unsteady lifting-line theory, and the lift coefficient from steady lifting-surface theory. Unfortunately, no unsteady lifting-surface calculations for an oscillating elliptic wing are presently available. We see that, in the limit of steady flow, agreement with steady lifting-surface results is quite good. Furthermore, with increasing k_0, the three-dimensional results approach their strip-theory values, as expected. Beyond $k_0 \approx 0.5$, however, this trend gradually reverses owing to the aforementioned divergence of the present theory at higher reduced frequencies. It is noteworthy that, beyond $k_0 \approx 0.5$, strip-theory results are quite adequate.

The lifting-line assumption that spanwise flow perturbations are small compared with those in planes normal to the span is violated near blunt wingtips, where the flow does not become two-dimensional no matter how large the aspect ratio. This gives rise to local regions of nonuniformity near the tips, the size of which is larger for blunter tips. The non-uniformities can be corrected by constructing additional asymptotic expansions valid in the immediate vicinity of the tips and matching them to the inner solution.

In steady flow, starting with the exact solution of Kinner (1937) for a circular wing, Jordan (1971*a*, *b*) carried out a detailed study of the flow field near a circular (or parabolic) wingtip. He found that, contrary to the classical assumption of (essentially) elliptic span loading, the actual loading contains a logarithmic term near the tip. As a consequence, the induced downwash contains a logarithmic singularity, which gives rise to an infinite upwash at the tip. Also, in relation to an oscillating rectangular wingtip, Landahl (1968) found a similar logarithmic term in the span-loading. It might be possible to derive similar results for an oscillating circular (or parabolic)

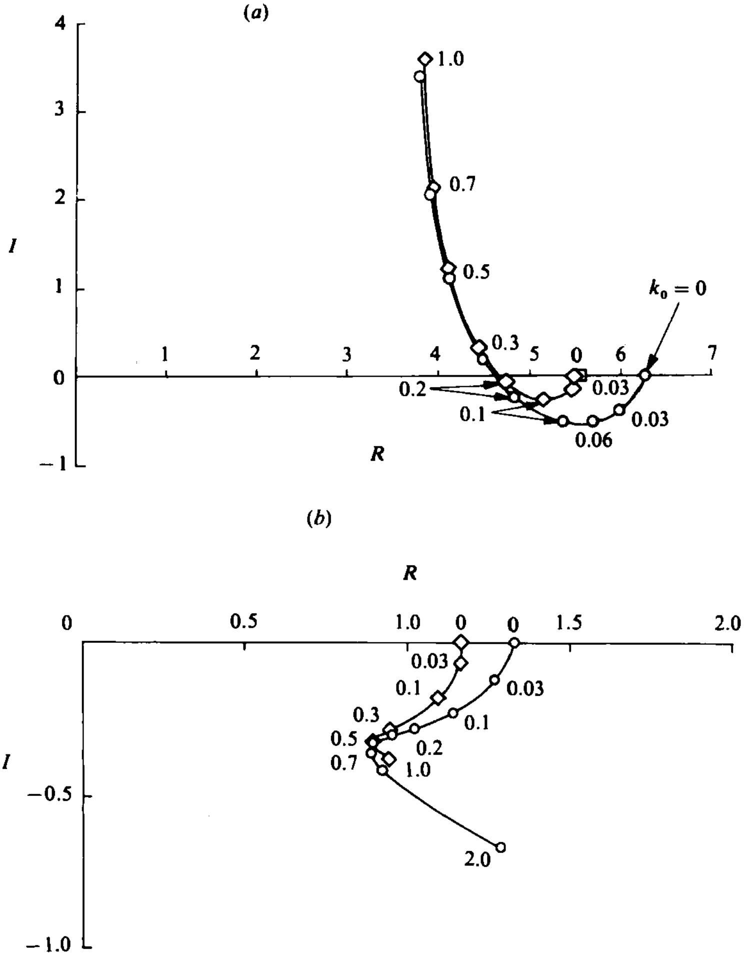

FIGURE 7. Complex vector diagram of $-\tilde{C}_{LP}$ and $-\tilde{C}_{MP}$ as functions of k_0 for an elliptic wing in pitch ($A = 16$): for legend see figure 6.

wingtip, using the exact solution of Schade & Krienes (1947) for an oscillating circular wing. Presumably, similar logarithmic terms in the span-loading and induced downwash would be uncovered.

For the present theory, the numerical results of Ahmadi (1980) for a family of wing planforms indicate that for elliptic and more slender planforms, the theory yields convergent total aerodynamic coefficients. We expect that for wings with blunter tips, such as the rectangular one, non-integrable singularities will show up at the tips.

9. Concluding remarks

Unsteady lifting-line theory has been developed for a wing that is completely flexible in the span direction. Wing displacements are prescribed, and the pressure throughout the flow field is determined in closed form to leading order in inverse aspect ratio. It is found that three-dimensional effects are manifested in the form of a convecting sinusoidal gust whose complex amplitude, to leading order, is a constant across the chord but varies across the span in a manner determined by the wing shape and motion. Compared with numerical lifting-surface theory, the theory reduces computation time significantly.

Using the present theory, Ahmadi & Widnall (1983) determined the influence of three-dimensionality on the energetic quantities – thrust, energy loss rate due to vortex shedding in the wake, power required to sustain the wing oscillations, and leading-edge suction force – for a finite wing oscillating in combined pitch and heave. Using these results, Ahmadi & Widnall have also determined the optimum motion of the wing – the motion which minimizes the energy loss rate for fixed prescribed total thrust. This is the three-dimensional counterpart of Wu's (1971*b*) study of the optimum motion of an airfoil in pitch and heave.

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